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Energy-Momentum Vector of The Classical Electron

Jeffrey M. Cohen[1] **and Errol Mustafa**[1]

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One of the problems often attributed to the classical electron is that energy and linear momentum do not transform as components of a 4-vector under Lorentz transformations. It is shown (with the example of an uncharged balloon) that this problem is not unique to the classical electron with its electromagnetic field extending to spatial infinity. For the balloon model and the classical electron it is shown that the cohesive surface stress makes a contribution to the energy and momentum in such a way that they transform as 4-vector components. From these and other considerations it is shown that the classical electron may be treated in a self-consistent manner.

1. INTRODUCTION

In flat space-time the energy and linear momentum of an isolated non-quantum mechanical system should transform as components of a 4-vector. In addition, for such a system to be in equilibrium, it is necessary that the internal forces and stresses within the body be balanced.

In this paper we investigate two systems which involve spherical shell mass distributions. We show that the inclusion of cohesive surface forces to give equilibrium leads naturally to the result that the energy and linear momentum transform as components of a 4-vector. This result is of importance in our discussion of the classical electron (Section 4), since one of the problems often attributed (Leighton, 1959) to models of the classical electron is that the energy and linear momentum do not transform as components of a 4-vector under Lorentz transformations. In particular, we discuss our results for the example of a charged mass shell (Section 4) in relation to the Abraham (1905)–Lorentz (1909)–Poincaré (1906) classical model for the electron. Furthermore, we compare our results with those of other authors (Kwal, 1949 Rohrlich, 1960; Fermi, 1922); we resolve the electron

[1]Physics Department, University of Pennsylvania, Philadelphia, Pennsylvania 19104.

self-energy problem, i.e., the energy and linear momentum remain finite in the point particle limit.

For completeness, we show in an Appendix that the models of Sections 3 and 4, together with the Poincaré classical electron, may be viewed as limiting cases of a more general system.

2. ENERGY-MOMENTUM AND CONSERVED QUANTITIES

Before proceeding with the details of our examples, we review (Cohen, 1968) the derivation of the expression for conserved quantities (such as the energy and linear momentum) generated by a non-quantum mechanical system. Use of this expression will be made in the following section. (The results obtained here are valid in general relativity and in the classical limit.)

In curvilinear coordinates, the conservation law

$$T^{\mu\nu}{}_{;\nu}=0 \tag{1}$$

yields conserved quantities for systems with symmetries (semicolon denotes covariant differentiation, which makes allowance for possible motion of the basis vectors). Symmetry-preserving transformations of the system are generated by Killing vectors ξ, with components ξ_μ, satisfying the equation $\xi_{\mu;\nu}+\xi_{\nu;\mu}=0$. Contracting the conservation law (1) with the Killing vector ξ yields

$$\xi_\nu T^{\mu\nu}{}_{;\nu}=(\xi_\mu T^{\mu\nu})_{;\nu}-\xi_{\mu;\nu}T^{\mu\nu}=(\xi_\mu T^{\mu\nu})_{;\nu} \tag{2}$$

because of the symmetry of $T^{\mu\nu}$ and the antisymmetry of $\xi_{\mu;\nu}$. Integrating over all space-time σ and applying the n-dimensional form (Synge, 1960) of Stokes' theorem yields

$$0=\int_\sigma (\xi_\mu T^{\mu\nu})_{;\nu}\,dV_4=\int_{\partial\sigma}\xi_\mu T^{\mu\nu}\,d\sigma_\nu \tag{3}$$

where $\partial\sigma$ denotes the boundary of σ. For sources that are bounded in space or fall off sufficiently rapidly at spatial infinity (as is the case for electromagnetic sources), equation (3) becomes the difference of two integrals evaluated over spacelike surfaces of constant time. Since this difference vanishes, the integral evaluated at constant time is independent of the spatial surface. Therefore the quantity

$$I=\int_{t\text{-const}}\xi_\mu T^{\nu 0}\,d\sigma_0 \tag{4}$$

is conserved, where $d\sigma_0$ is the volume element in 3-space.

In flat space-time using Cartesian coordinates ($x^0=t, x^1=x, x^2=y, x^3=z$) the Killing vector associated with stationarity is $\xi=\partial_t$ (i.e., $\xi_0=1$). Thus, the total energy E, the conserved quantity associated with time translation invariance, is

$$E = \int_{t\text{-const}} T^{00}\, d\sigma_0 \tag{5}$$

Similarly, the conserved quantity associated with space translation invariance, (e.g., in the z direction and generated by $\xi = \partial_z$) is the linear momentum

$$P^z = \int_{t\text{-const}} T^{30}\, d\sigma_0 \tag{6}$$

3. BALLOON MODEL

The system considered here consists of a gas-filled balloon moving with velocity v along the z axis with respect to a laboratory frame of reference L'. The balloon is a sphere of radius r_0 in its rest frame L. The mass distribution in the infinitesimally thin surface of the balloon is specified by

$$\rho_{\text{surf}} = m\delta(r - r_0) \tag{7}$$

where $\int d^3r\, \delta(r - r_0) = 1$. The pressure P and gas density ρ are taken to be uniform within the balloon's interior.

The outward-acting pressure is balanced by a surface tension (tangential stress in the surface of the balloon) given by

$$S_\delta = S\delta(r - r_0) \tag{8}$$

It can be shown (e.g., by a free body diagram) that equilibrium requires

$$S = -2\pi r_0^3 P \tag{9}$$

The stress-energy tensor may be written as

$$T^{\mu\nu} = (\rho_{\text{surf}} + \rho) U^\mu U^\nu + P(U^\mu U^\nu + \eta^{\mu\nu}) + T^{\mu\nu}_{\text{cohes}} \tag{10}$$

where U^μ is the 4-velocity of the balloon and $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. (It may be of interest to note that the tensor $U^\mu U^\nu + \eta^{\mu\nu}$ projects any 4-vector onto the space perpendicular to U^μ.) With respect to spherical coordinates $(x^0 = t,\ x^1 = \hat{r},\ x^2 = r\hat{\theta},\ x^3 = r\sin\theta\,\hat{\phi})$ in L one observes that

$$T^{\mu\nu}_{\text{cohes}} = S_\delta(\delta^\mu_2 \delta^\nu_2 + \delta^\mu_3 \delta^\nu_3) \tag{11}$$

In order to calculate the energy and linear momentum of the system in the laboratory frame L' as well as the rest frame L, one could evaluate the relevant components of equation (10) in L'; to make use of the expressions for energy and linear momentum [equations (5) and (6)] $T^{\mu\nu}$

shall be expressed in Cartesian coordinates. The first terms on the rhs of (10) can be transformed using the properties: U^μ is a 4-vector, and $\eta^{\mu\nu}$ is the same in all frames. In L', $U^\mu = \gamma(1, 0, 0, v)$ with respect to Cartesian coordinates, where $\gamma \equiv (1-v^2)^{-1/2}$. Denoting the sum of all but the last term on the rhs of (10) by $T^{\mu\nu}_{\text{matter}}$, one obtains

$$\begin{aligned} T^{00}_{\text{matter}} &= \gamma^2(\rho_{\text{surf}}+\rho)+P(\gamma^2-1) \\ T^{03}_{\text{matter}} &= \gamma^2 v(\rho_{\text{surf}}+P+\rho) \end{aligned} \tag{12}$$

Before proceeding with the evaluation of the integral expressions (5) and (6) for the energy and z momentum, it should be noted that these integrals are more conveniently evaluated with respect to coordinates in the rest frame rather than the laboratory frame. In terms of the unprimed coordinates in L, (5) and (6) may be rewritten

$$E = \gamma^{-1} \int_{t\text{-const}} T^{00}\, d\sigma_L \tag{13}$$

$$P^z = \gamma^{-1} \int_{t\text{-const}} T^{03}\, d\sigma_L \tag{14}$$

where the volume elements in L and L' are related by $d\sigma_L = \gamma\, d\sigma_{L'}$. If the contributions to the total energy and linear momentum (in the laboratory frame) that involve $T^{\mu\nu}_{\text{matter}}$ are called E_{matter} and P^z_{matter}, respectively, equations (12)-(14) yield

$$\begin{aligned} E_{\text{matter}} &= \gamma\left(m+m'+\frac{4\pi r_0^3}{3}Pv^2\right) \\ P^z_{\text{matter}} &= \gamma v\left(m+m'+\frac{4\pi r_0^3}{3}P\right) \end{aligned} \tag{15}$$

where $\gamma^2-1=\gamma^2v^2$ has been used, and $m'=\frac{4}{3}\pi r_0^3\rho$ is the rest mass of the interior gas. It is apparent from (15) that E_{matter} and P^z_{matter} do not by themselves constitute a 4-vector. Since one expects the total energy and linear momentum of the system to form a 4-vector, there should be additional contributions to the energy and momentum. In this model these contributions (denoted by E_{cohes} and P^z_{cohes}) arise from the cohesive stress.

In order to transform $T^{\mu\nu}_{\text{cohes}}$ (and thus evaluate E_{cohes} and P^z_{cohes} in L'), one forms the tensor

$$T_{\text{cohes}} = T_{\mu\nu\,\text{cohes}}\, \omega^\mu \otimes \omega^\nu \tag{16}$$

where the ω^μ form a basis of one-forms and $\otimes$ is the Cartesian product. In L we choose a basis of spherical one-forms, $\omega^0 = dt$, $\omega^1 = dr$, $\omega^2 = r\, d\theta$,

$\omega^3 = r \sin\theta\, d\phi$. From equations (11) and (16) we find

$$T_{\text{cohes}} = S_\delta (\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3) \tag{17}$$

In L, ω^2 and ω^3 may be expressed in terms of Cartesian one-forms as follows:

$$\begin{aligned} \omega^2 &= \cos\theta \cos\phi\, dx + \cos\theta \sin\phi\, dy - \sin\theta\, dz \\ \omega^3 &= \cos\phi\, dy - \sin\phi\, dx \end{aligned} \tag{18}$$

To transform to the laboratory frame L', we make use of the Lorentz transformation

$$\begin{aligned} dt &= \gamma(dt' - v\, dz') \\ dx &= dx' \\ dy &= dy' \\ dz &= \gamma(dz' - v\, dt') \end{aligned} \tag{19}$$

where primed coordinates are those used in L'.

From equations (18) and (19) we find

$$\begin{aligned} \omega^2 &= \cos\theta \cos\phi\, dx' + \cos\theta \sin\phi\, dy' - \gamma \sin\theta\, (dz' - v\, dt') \\ \omega^3 &= \cos\phi\, dy' - \sin\phi\, dx' \end{aligned} \tag{20}$$

From equations (16), (17) and (20) and using a basis of Cartesian one-forms $dx^{\mu'}$ in L', we identify

$$T^{00}_{\text{cohes}} = \gamma^2 v^2 \sin^2\theta\, S_\delta, \qquad T^{03}_{\text{cohes}} = \gamma^2 v \sin^2\theta\, S_\delta \tag{21}$$

in the laboratory frame. Equations (13), (14), and (21) give

$$E_{\text{cohes}} = \tfrac{2}{3}\gamma v^2 S, \qquad P^z_{\text{cohes}} = \tfrac{2}{3}\gamma v S \tag{22}$$

As above for E_{matter} and P^z_{matter}, E_{cohes} and P^z_{cohes} do not transform individually as components of a 4-vector. The total energy E and linear momentum P^z are found by adding the two contributions (15) and (22). One obtains

$$\begin{aligned} E &= \gamma(m + m') = \gamma M \\ P^z &= \gamma(m + m')v = \gamma M v \end{aligned} \tag{23}$$

where $m + m'$ represents the total rest mass M of the system. The terms in P and S in (15) and (22) cancel when added because of equation (9). For translation in the z direction, P^z is the only nonzero component of linear momentum. It is then clear from (23) that the total energy E and linear momentum P^z transform as components of a 4-vector, as hoped. It is interesting to note that the requirement that the system be in equilibrium is sufficient to ensure that the energy and linear momentum transform as components of a 4-vector.

The balloon model is of importance since it provides an example of a completely mechanical system for which the noncohesive contributions to the energy and linear momentum do not by themselves constitute a 4-vector. The same problem arises in models of the classical electron (Section 4), and might be attributed falsely to the infinite spatial extent of the electromagnetic fields. The balloon model (with no charge, and hence no electromagnetic fields) demonstrates that the nature of these fields is not the cause of the problem.

4. THE CLASSICAL ELECTRON

Historically, the electron was first thought of as a rigid, charged sphere of small but nonzero radius. This classical model was introduced by Abraham (1905) and developed by Abraham and Lorentz (1909) in the early part of this century. The Lorentz shell model has the drawback that the electromagnetic forces that tend to explode the electron are not balanced or compensated. In addition, the energy and linear momentum do not transform as components of a 4-vector (Rohrlich, 1960; Pais, 1948). Poincaré (1906) attempted to resolve these problems by postulating the existence of a nonelectromagnetic cohesive force (Poincaré stress) within the electron. This took the form of a negative pressure in the electron's interior. For a system with density ρ and pressure P, the nonelectromagnetic contributions Bergmann (1942) to $T^{\mu\nu}$ are $\rho U^\mu U^\nu + P(U^\mu U^\nu + \eta^{\mu\nu})$. Poincaré, however, omitted the $PU^\mu U^\nu$ term. This may have led others to remark that the nature and transformation properties of the Poincaré stress are not well formulated (Leighton, 1949). In addition, all these models retain the problem of an infinite electromagnetic self-energy in the limit of a (zero-radius) point particle.

In this section we present a shell model for the classical electron which removes the above difficulties encountered by earlier models.

Consider a system consisting of a uniformly charged shell of charge q and total mass m (with $q \gg m$ in dimensionless units) moving with velocity v along the z axis in the laboratory frame L'. As before (Section 3), the shell is a sphere of radius r_0 in the rest frame L.

The matter distribution is most conveniently described in L. It is

$$\rho = \kappa\delta(r - r_0) \tag{24}$$

where κ depends on q, m, and r_0.

The stress-energy tensor of such a system is

$$T^{\mu\nu} = \rho U^\mu U^\nu + \frac{1}{4\pi}(F^\nu_{\ \alpha} F^{\mu\alpha} - \tfrac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}) + T^{\mu\nu}_{\text{cohes}} \tag{25}$$

where $g^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ in Cartesian or spherical coordinates, $F^{\mu\nu}$ is the Maxwell tensor, and U^μ is the 4-velocity of the shell. $T^{\mu\nu}_{\text{cohes}}$ represents a tangential cohesive surface stress in the shell which balances the electromagnetic repulsion within the shell. Thus, the surface stress, as in the previous example, acts as a glue. $T^{\mu\nu}_{\text{cohes}}$ is given by (8) and (11) of Section 3, with equilibrium of the internal cohesive and repulsive forces requiring

$$S = -\frac{q^2}{4r_0} \tag{26}$$

for $q \gg m$.

The first two terms on the rhs of equation (25) are identified as the contributions of the matter density and the electromagnetic field (Wheeler, 1962), respectively, to $T^{\mu\nu}$. We may thus write

$$\begin{aligned} T^{\mu\nu}_{\text{matter}} &= \rho U^\mu U^\nu \\ T^{\mu\nu}_{\text{em}} &= \frac{1}{4\pi}(F^\nu_{\ \alpha} F^{\mu\alpha} - \tfrac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}) \end{aligned} \tag{27}$$

It is instructive to calculate the individual contributions of the matter density, em field, and cohesive stress to the total energy-momentum 4-vector (evaluated in the laboratory frame). We show that although the individual contributions from the em field and cohesive stress to the total energy-momentum are not themselves 4-vectors, the sum of all contributions is a 4-vector.

If one considers only the contribution of $T^{\mu\nu}_{\text{matter}}$ to $T^{\mu\nu}$ and denotes the corresponding contributions to the total energy and linear momentum in L' by E_{matter} and P^z_{matter}, respectively, one finds

$$\begin{aligned} E_{\text{matter}} &= \gamma\kappa \\ P^z_{\text{matter}} &= \gamma\kappa v \end{aligned} \tag{28}$$

where use has been made of equations (13), (14), (24), and (27). One observes from (28) that E_{matter} and P^z_{matter} form a 4-vector. Integration of the mass equation $dm/dr = 4\pi r^2 T^{00}$ gives the following expression for κ to first order in v:

$$\kappa = m - \frac{q^2}{2r_0} \tag{29}$$

The components of $T^{\mu\nu}_{\text{em}}$ in L' may be found in a manner analogous to that used in Section 3 for $T^{\mu\nu}_{\text{cohes}}$: One first calculates in the rest frame L the tensor

$$F = \tfrac{1}{2} F_{\mu\nu} \omega^{\mu} \otimes \omega^{\nu} \tag{30}$$

Using equations (20) and (30), one may identify $F_{\mu\nu}$ (and therefore $T^{\mu\nu}$) in L'. Denoting the contributions to the total energy and linear momentum in L' by E_{em} and P^{z}_{em}, respectively, one obtains [using (13) and (14)] the well-known results (Leighton, 1959)[1]

$$\begin{aligned} E_{\text{em}} &= \frac{\gamma q^2}{2r_0}\left(1+\frac{v^2}{3}\right) \\ P^{z}_{\text{em}} &= \frac{2\gamma q^2 v}{3r_0} \end{aligned} \tag{31}$$

It is clear from the above that E_{em} and P^{z}_{em} do not form components of a 4-vector. To circumvent this difficulty, some authors (Kwal, 1949; Rohlich, 1960; Jackson, 1975) have modified the definitions of the total energy and linear momentum [equations (5) and (6)] in such a way that E_{em} and P^{z}_{em} transform as components of a 4-vector. [This may be achieved (Kwal, 1949; Rohrlich, 1960; Jackson, 1975), for example, by the introduction of a covariant element of 4-volume $d\sigma^{\mu}$.] Not only are such changes in the definitions somewhat arbitrary, but they are also unnecessary: Physically, what is required is that the total energy and linear momentum of a system form a 4-vector. This may be achieved using the standard definitions of energy and linear momentum adopted here, as we show below.

As before (Section 3), the relevant components of $T^{\mu\nu}_{\text{cohes}}$ are given by (21), where equations (8) and (26) specify S_{δ}. The contributions of $T^{\mu\nu}_{\text{cohes}}$ to the energy E_{cohes} and linear momentum P^{z}_{cohes} are

$$\begin{aligned} E_{\text{cohes}} &= -\gamma q^2 v^2/6r_0 \\ P^{z}_{\text{cohes}} &= -\gamma q^2 v/6r_0 \end{aligned} \tag{32}$$

From (32) it can be seen that E_{cohes} and P^{z}_{cohes} do not make up the timelike and spacelike components of a 4-vector. Adding the rhs of (31) to the rhs of (32) yields

$$\begin{aligned} E_{\text{em}} + E_{\text{cohes}} &= \gamma q^2/2r_0 \\ P^{z}_{\text{em}} + P^{z}_{\text{cohes}} &= \gamma q^2 v/2r_0 \end{aligned} \tag{33}$$

[1]Note that $\gamma q^2/2r_0$ is commonly referred to as the self-energy (of the em field).

From (31) one observes that the sum of the cohesive stress energy-momentum and the em energy-momentum constitute a 4-vector, as mentioned above.

The total energy E and linear momentum P^z are found by adding the three contributions (28), (31), and (32). One obtains

$$\begin{aligned} E &= \gamma m \\ P^z &= \gamma m v \end{aligned} \tag{34}$$

where use has been made of (29). From (34) it is manifest that (E, P^z) constitutes a 4-vector. Further, the self-energy of the em field does not appear in this final expression. Thus, the point-particle limit may be taken without incurring either an infinite total energy or linear momentum.

5. CONCLUDING REMARKS

There are systems which reqire the existence of cohesive stresses for equilibrium. These stresses contribute to the energy and linear momentum in such a way that the total energy-momentum is a 4-vector. Our definitions of energy and linear momentum are the standard ones.

We have also discussed the classical electron, modeled as a charged mass shell. A surface stress holds the electron in equilibrium and contributes to the total energy and linear momentum such that they transform as components of a 4-vector. These features were absent from the original Abraham–Lorentz model for the electron. Poincaré was only partially successful in incorporating these features in his model for the electron (see above). Moreover, the electron self-energy diverges in the point-particle limit for the Poincaré model. Our treatment of the electron does not have this problem, since the total energy is finite and independent of the electron radius. Thus, we have shown that it is possible to treat the classical electron in a self-consistent manner.

These examples illustrate the importance of including all contributions of the stress-energy tensor when considering the equilibrium and Lorentz transformation properties of classical bodies. When this is done, the energy and linear momentum naturally constitute components of a 4-vector.

APPENDIX: THE CHARGED BALLOON

The systems described in Sections 3 and 4, together with the Poincaré classical electron, are each special cases of a more general system: a charged balloon supported by a cohesive surface stress and/or an interior pressure. Using the notation adopted in Sections 3 and 4 and adding contributions from (15), (22), and (31), one finds that the total energy and linear momentum of the system are

$$E = \gamma\left[M + \frac{v^2}{3}\left(4\pi r_0^3 P + 2S + \frac{q^2}{2r_0}\right)\right]$$
$$P^z = \gamma\left[M + \tfrac{1}{3}\left(4\pi r_0^3 P + 2S + \frac{q^2}{2r_0}\right)\right]v \tag{A1}$$

where the total mass M is found (from the mass equation) to be

$$M = \kappa + \frac{4\pi r_0^3 \rho}{3} + \frac{q^2}{2r_0} \tag{A2}$$

to first order in v.

If the condition that the body be in equilibrium,

$$4\pi r_0^3 P + 2S + \frac{q^2}{2r_0} = 0 \tag{A3}$$

is met, one observes from (A1) that the energy and linear momentum constitute a 4-vector. One sees that for the case $q=0$, (A3) reduces to (9), (A1) reduces to (23), and the balloon model is recovered. Similarly, for the case $P=0$, (A3) reduces to (26), (A1) reduces to (34) (if $\rho=0$), and the model of a charged mass shell is recovered. In the absence of a cohesive surface stress ($S=0$), (A3) reduces to

$$P = -q^2/8\pi r_0^4 \tag{A4}$$

It may be seen from (A4) that if $S=0$, the shell can be supported by a negative internal pressure, as Poincaré attempted to show (cf. Section 4 for further discussion).

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